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ON THE COMPLEXITY OF THE PLANAR EDGE-DISJOINT PATHS PROBLEM WITH TERMINALS ON THE OUTER BOUNDARY

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It is shown that both the undirected and the directed edge-disjoint paths problem are NP-complete, if the supply graph is planar and all edges of the demand graph are incident with vertices lying on the outer boundary of the supply graph. In the directed case, the problem remains NP-complete, if in addition the supply graph is acyclic. The undirected case solves open problem no. 56 of A. Schrijver's book *Combinatorial Optimization*.

1. Introduction

Let $G=(V,E)$ and $H=(T,R)$ be both directed or undirected graphs with $T\subseteq V$ and $E\cap R=\emptyset$. The graphs may have parallel edges, but for convenience it is assumed that they don't have loops. G is called the *supply graph* and H is called the *demand graph*. The edges $R=\{(s_1,t_1),(s_2,t_2),\dots,(s_n,t_n)\}$ of H are named *nets*, and T is the set of *terminals*. The pair (G,H) forms an instance of the *directed* resp. *undirected edge-disjoint paths problem*. It has a solution if and only if there exist edge-disjoint s_it_i–paths for $i=1,\dots,n$.

An undirected graph is called *Eulerian*, if an even number of edges are incident with every vertex, and a directed graph is called *Eulerian*, if at every vertex the number of incoming edges equals the number of outgoing edges.

The graph $G+H$ denotes the graph with vertex set V whose edge set is the union of E and R. For $U\subseteq V$, let $\delta(U)\subseteq E$ be the set of edges with one endpoint in U and the other one in $V-U$. We say that the undirected

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edge-disjoint paths problem (G, H) fulfills the *cut condition*, if and only if

$$|\delta(U) \cap R| \leq |\delta(U) \cap E| \text{ for all } U \subseteq V. \tag{1}$$

The cut $\delta(U)$ is called *tight*, if equality holds in (1).

The undirected as well as the directed edge-disjoint paths problem is NP-complete in the general case, and also, if the supply graph G is planar [2]. Both problems remain NP-complete if $G+H$ is planar. This has been proven by Middendorf and Pfeiffer [3] for the undirected case; the directed case follows by a straightforward tranformation, cf. [6].

For the directed case, Vygen showed in [6]:

Theorem 1. *The directed edge-disjoint paths problem is NP-complete, even if the supply graph G is planar and acyclic.*

On the other hand, a famous theorem of Okamura and Seymour [4] states that under certain conditions the undirected edge-disjoint paths problem becomes polynomial-time solvable:

Theorem 2. *Let G be planar, $G+H$ Eulerian, and let all nets be incident with vertices on the boundary of the outer face of G. Then the undirected edge-disjoint paths problem (G, H) has a solution if and only if the cut condition holds.*

The undirected edge disjoint paths problem has a good characterization even if the Eulericity condition in the theorem of Okamura and Seymour is weakened. If the vertices on the outer face are not supposed to have even degree the problem is still solvable in polynomial time, as was proven by Frank [1].

We show that Eulericity cannot be omitted completely:

Theorem 3. *The undirected edge-disjoint paths problem is NP-complete, even if the supply graph G is planar and all terminals lie on the boundary of the outer face of G.*

This answers Problem 56 of Schrijver [5].

For directed graphs a similar result holds, even if G is acyclic:

Theorem 4. *The directed edge-disjoint paths problem is NP-complete, even if the supply graph G is planar and acyclic and all terminals lie on the boundary of the outer face of G.*

This strengthens Theorem 1.

2. Proofs

Proof of Theorem 3. The proof is by reduction from SATISFIABILITY.

Let $\mathcal{F}$ be a Boolean formula in conjunctive normal form, consisting of literals $\{x_1, x_2, \ldots, x_k\} \cup \{\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_k\}$ and clauses $C_1, C_2, \ldots, C_m$. We may assume that each clause is non-empty and does not contain both a variable x_i and its negation $\bar{x}_i$. Formula $\mathcal{F}$ is *true* if and only if every clause contains at least one *true* literal.

Based on $\mathcal{F}$, an undirected edge-disjoint paths problem is constructed as follows: The demand graph H consists of edges $v_i w_i$ for every variable x_i $(i = 1, \ldots, k)$ and of edges $s_j t_j$ and $s'_j t'_j$ for every clause C_j $(j = 1, \ldots, m)$. The supply graph G is a rectangular structure as described in Figure 1.

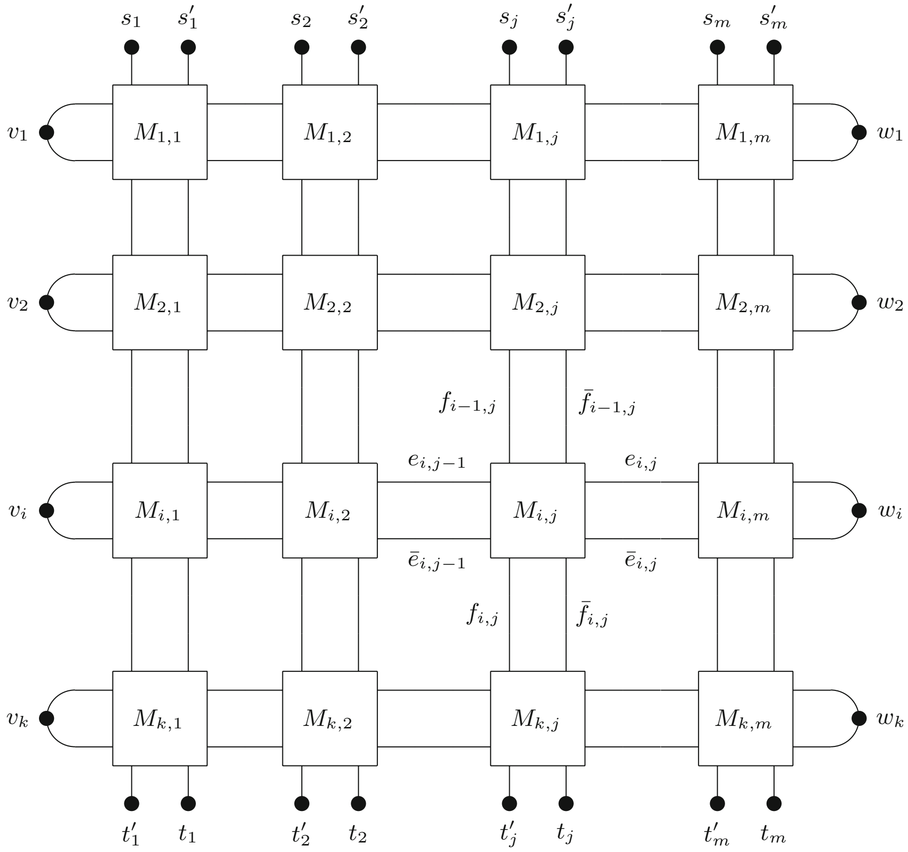

Figure 1. $G + H$

The substructures $M_{i,j}$ are special subgraphs, which are connected to the neighboring special subgraphs and to the terminals by eight edges. These edges altogether are denoted (see Figure 1) by $e_{i,j}$, $\bar{e}_{i,j}$ $(i = 1,\dots,k,\ j = 0,\dots,m)$ and $f_{i,j}$, $\bar{f}_{i,j}$ $(i = 0,\dots,k,\ j = 1,\dots,m)$. There are three types of such special subgraphs; $M_{i,j}$ is of

Type X if C_j contains literal x_i,
Type $\bar{X}$ if C_j contains literal $\bar{x}_i$,
Type 0 if C_j contains neither x_i nor $\bar{x}_i$.

The three in fact very simple types of subgraphs are depicted in Figure 2.

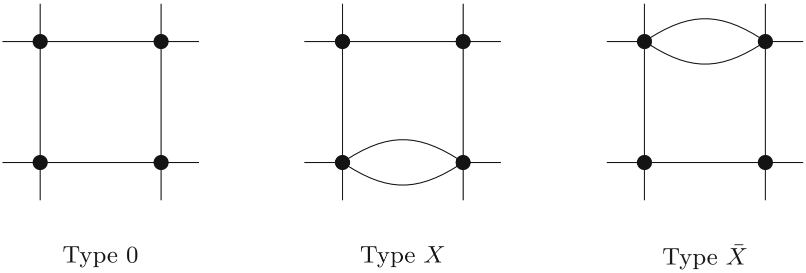

Figure 2. Special subgraphs

Claim 1. *In a solution to the edge-disjoint paths problem (G,H), all vertical edges of G are used by the s_jt_j- and $s'_jt'_j$-paths. (The terms* vertical *and* horizontal *refer to Figures 1 and 2.)*

This is obvious, as all horizontal cuts are tight.

Claim 2. *Every v_iw_i-path of a solution uses either the "upper" edges $e_{i,0}, e_{i,1},\dots,e_{i,m}$ or the "lower" edges $\bar{e}_{i,0},\bar{e}_{i,1},\dots,\bar{e}_{i,m}$.*

This follows from Claim 1, because no vertical edges remain for switching between the "upper" and the "lower" path.

Claim 3. *No s_lt_l- or $s'_lt'_l$-path of a solution uses a (horizontal) edge $e_{i,j}$ or $\bar{e}_{i,j}$ $(i=1,\dots,k,\ j=0,\dots,m)$.*

This is clear for $j=0$. Let $e=e_{i,j}$ be an edge used by an s_lt_l- or $s'_lt'_l$-path. Choose $j \geq 1$ minimal and consider special subgraph $M_{i,j}$. From the edges connecting it to the remaining graph, the four vertical ones ($f_{i-1,j}$, $f_{i,j}$, $\bar{f}_{i-1,j}$, $\bar{f}_{i,j}$) are always used by solution paths (Claim 1). The two horizontal edges $\bar{e}_{i,j-1}$ and $\bar{e}_{i,j}$ are used by the v_iw_i-path (Claim 2). Edge $e_{i,j-1}$ is not contained in any solution path, by the minimality of j. So seven out of the eight edges are used by solution paths, which is impossible. (The case $e = \bar{e}_{i,j}$ can be proven in the same way.)

Claim 4. *An s_jt_j-path of a solution may cross the corresponding $s'_jt'_j$-path only in a special subgraph of type X or of type $\bar{X}$.*

This follows from Claim 3, and from the fact that crossing is not possible in a special subgraph of type 0, because there is just one unused horizontal edge available.

Clearly, if the s_jt_j- and $s'_jt'_j$-path cross in a subgraph of type X or $\bar{X}$, they have to use the two parallel edges.

Claim 5. *The Boolean formula $\mathcal{F}$ is satisfiable if and only if the edge-disjoint paths problem (G,H) has a solution.*

Assume that there is an assignment $\{x_1, x_2, \ldots, x_k\} \to \{\mathit{true},\ \mathit{false}\}$ that satisfies $\mathcal{F}$. Select the v_iw_i-paths accordingly: let them be "upper paths" for $x_i = \mathit{true}$ and "lower paths" otherwise. Every clause C_j contains at least one true literal x_i or $\bar{x}_i$. Select path s_jt_j and $s'_jt'_j$ such that they cross exactly once, namely in special subgraph $M_{i,j}$.

Conversely, if (G,H) has a solution, then let x_i be *true* if the v_iw_i-path goes the "upper way", and $x_i = \mathit{false}$ otherwise (cf. Claim 2). G is planar, so for every j, the paths s_jt_j and $s'_jt'_j$ belonging to clause C_j have at least one common node, i.e. they cross somewhere. By Claim 4 they do it in a subgraph $M_{i,j}$ of type X or $\bar{X}$. Then C_j contains literal x_i or $\bar{x}_i$. In the first case, i.e. if $M_{i,j}$ is of type X, the v_iw_i-path goes the "upper way", x_i is *true* and C_j is satisfied. Similarly, if C_j contains literal $\bar{x}_i$.

The supply graph G is planar, all terminals are on the boundary of the outer face of G, and the size of $G+H$ is clearly polynomially bounded by the size of $\mathcal{F}$. This completes the proof of the theorem. ∎

Proof of Theorem 4. Direct all edges of the supply graph G used in the proof of Theorem 3 and depicted in Figure 1 from left to right and from top to bottom, and replace the special subgraphs $M_{i,j}$ by their directed counterparts depicted in Figure 3.

The resulting supply graph is acyclic, and the proof of Theorem 3 may be carried over almost one-to-one. (Claim 3 is trivial in the directed case and may be omitted.) ∎

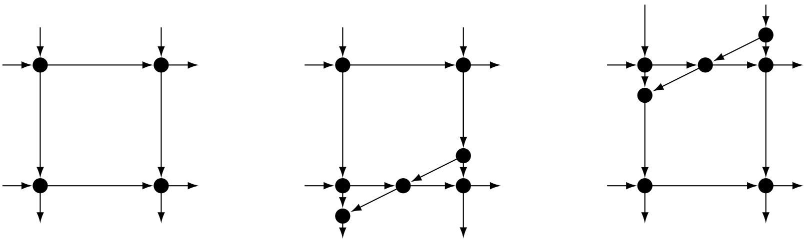

Figure 3. Directed special subgraphs

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